Eigenvalues and Eigenfunctions Differential Equations X. Du

- Suppose that we have some linear differential operator D on some function f such that Df is a scalar multiple of f. Then f is an **eigenfunction** of D and the scalar multiple is an **eigenvalue**.
  - i.e. $Df = \lambda f$, where f is an eigenfunction and λ is its associated eigenvalue.
  - An eigenfunction is a type of eigenvector.
  - By convention, f is non-zero. $f(x) = 0$ is trivial as this is true for any D.
  - Allowable eigenvalues and eigenfunctions are usually restricted by boundary conditions.
- Why do we care about this?
  - Eigenvalue and eigenfunction analysis are important to the study of boundary value problems (BVPs), as they often express what a differential equation or its solutions are allowed to be given boundary conditions. This is especially important for partial differential equations (PDEs) with BVPs.
- Applications:
  - Vibrating strings. The ends of the string are fixed, so BVP.
  - Thermodynamics. The temperature at the ends of a rod is usually held constant.
  - Signals and systems
  - Quantum physics
    - Schrödinger equation: time-independent version takes the form $Df = \lambda f$
    - Particle in a box: the ends of the box are fixed
- How to solve linear ordinary differential operator D. If D is ordinary:
  - Substitute into $Df - \lambda f = 0$ to get a linear homogeneous ODE.
  - Obtain the characteristic polynomial and solve.
  - Use superposition to obtain the general solution.
  - Substitute boundary conditions into general solution.
  - Solve for as many unknowns as you can.
  - Done.
  - Compare this process to eigenvector analysis of matrices.